Self-similarity in Newtonian Cosmology

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Motivation

- The properties and existence of the dark matter is one of the most fascinating questions in cosmology.
- We present a **dark fluid model** described as a non-relativistic and self-gravitating fluid
- We studied these coupled **non-linear differential equation** systems using self-similar time-dependent solutions
- Our main goal of this research is to find **scaling solutions** of the gravitational fields, which can be good candidates to describe the evolution of the Universe or collapse of compact astrophysical objects

The Model

- Continuity Equation

$$\partial_t \rho + \mathrm{div}(\rho \boldsymbol{u}) = 0,$$

$$\rho \partial_t \boldsymbol{u} + \mathrm{div}(\rho \boldsymbol{u} \otimes \boldsymbol{u}) = -\nabla P(\rho) - \rho \nabla \Phi + \rho \boldsymbol{g}$$

$$\nabla^2 \Phi = 4\pi G \rho$$

$$P = P(\rho)$$

- Continuity Equation
- Euler Equation

$$\partial_t \rho + \operatorname{div}(\rho \boldsymbol{u}) = 0,$$

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- Poisson Equation

- Continuity Equation
- Euler Equation

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$$\rho \partial_t \boldsymbol{u} + \mathrm{div}(\rho \boldsymbol{u} \otimes \boldsymbol{u}) = -\nabla P(\rho) - \rho \nabla \Phi + \rho \boldsymbol{g}$$

$$\nabla^2 \Phi = 4\pi G \rho$$

$$P = P(\rho)$$

- Poisson Equation
- Equation of State

- We used polytropic EoS:

$$P(\rho) = w\rho^n, \qquad \text{where } n = 1$$

- Dark Fluid: $w = -1$
- Momentum conservation:

$$\nabla P(\rho) + \rho \nabla \Phi = 0$$

- Rotation:

$$\rho \boldsymbol{g} = \frac{\rho \sin\theta \omega^2 r}{t^2} \quad \omega : \text{angular velocity}$$

- Rotation is slow! $\Rightarrow$ Spherically symmetry is not broken
- Spherical Symmetry:

$$\partial_t \rho + (\partial_r \rho) u + (\partial_r u)\rho + \frac{2u\rho}{r} = 0,$$

$$\partial_t u + (u\partial_r) u = -\frac{1}{\rho}\partial_r P - \nabla\Phi + \frac{\sin\theta\omega^2 r}{t^2},$$

$$\Delta\Phi = 4\pi G\rho$$

$$P = P(\rho) \ .$$

- Self-similarity in 1D $\Rightarrow$ Sedov-Taylor Ansatz
  G. I. Taylor, British Report RC-210, June 27, 1941.
  IF Barna, MA Pocsai, GG Barnaföldi Mathematics 10 (18), 3220

$$u(r,t) = t^{-\alpha} f\left(\frac{r}{t^{\beta}}\right) \quad \rho(r,t) = t^{-\gamma} g\left(\frac{r}{t^{\beta}}\right)$$
$$\Phi(r,t) = t^{-\delta} h\left(\frac{r}{t^{\beta}}\right),$$

- (f, g, h) shape-functions only depend on $\zeta = rt^{-\beta}$
- $\alpha, \beta, \gamma, \delta$ similarity exponents
- The β describes the rate of spread of the spatial distribution
- Other exponents describe the rate of decay of the intensity of the corresponding field

- Self-Similarity: PDE reduce to ODE
- Depend only on ζ self-similar variable
- Algebraic equation system for the exponents
  $\Rightarrow \alpha = 0$, $\beta = 1$, $\gamma = 2$, and $\delta = 0$

$$-\zeta g'(\zeta) + f'(\zeta)g(\zeta) + f(\zeta)g'(\zeta) + \frac{2f(\zeta)g(\zeta)}{\zeta} = 0,$$

$$-\zeta^2 f'(\zeta) + \zeta f'(\zeta)f(\zeta) + \frac{wg'(\zeta)}{g(\zeta)} = -h'(\zeta)\zeta + \omega^2 \sin\theta \zeta^2,$$

$$h'(\zeta) + h''(\zeta)\zeta = g(\zeta) 4\pi G\zeta \ .$$

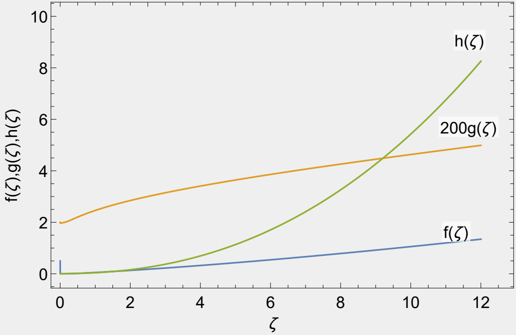
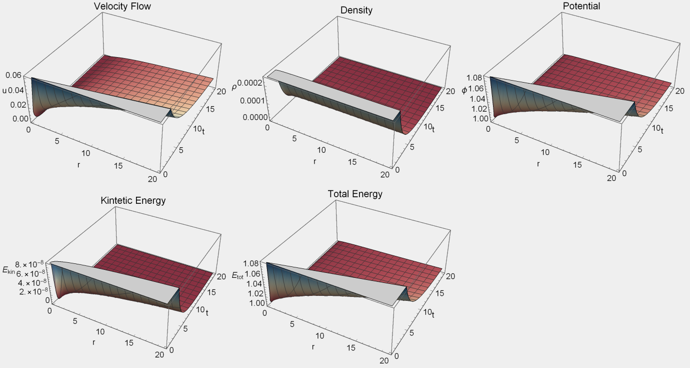


Figure: Numerical solutions of the shape functions, the integration was started at $\zeta_0 = 0.001$, and the initial conditions of $f(\zeta_0) = 0.5$, $g(\zeta_0) = 0.01$, $h(\zeta_0) = 0$, and $h'(\zeta_0) = 1$ were used. For the better visibility function $g(\zeta)$ was scaled up with a factor of 200. The values are given in geometrized units.

Dynamical Variables

To investigate fluid dynamics in time and space to understand general trends or physical phenomena as the function of the initial conditions.

$$\epsilon_{kin}(r,t) = \frac{1}{2}\rho(r,t)u^2(r,t), \quad \epsilon_{tot}(r,t) = \epsilon_{kin}(r,t) + \Phi(r,t).$$



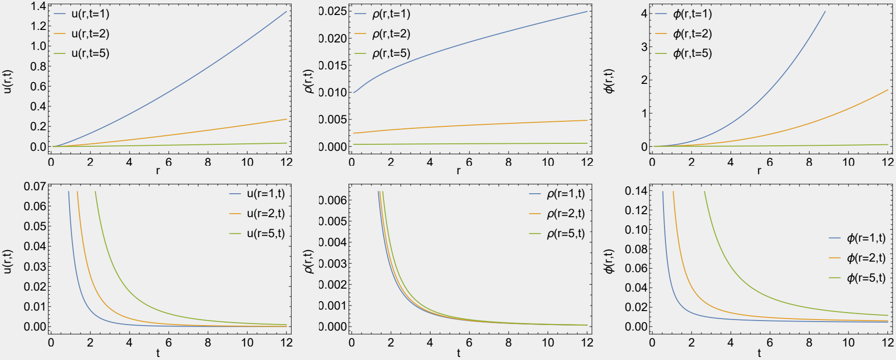


Figure: Different radial (1st row) and time (2nd row) projections of the velocity flow, density, and gravitational potential density for the non-rotating case

Connection to Friedmann Equation

- We are introducing a well-known scale-factor $a(t)$ which contains all of the temporal changes
- Relative distances in time: $R(t) = a(t)l$
- $\Omega(t) \subset \mathbb{R}^3$ is a sphere with radius $R(t)$ and $r \in (0, R(t))$

Mass

$$M(t) = \int_{\Omega(t)} \rho(R(t), t)dV = 4\pi \int_0^r \rho(R(t), t)R(t)^2 dR(t)$$

Mass Conservation

$$\frac{d}{dt}M(t) = 4\pi \frac{d}{dt} \int \rho(a(t)l, t)a^3(t)l^2 dl \stackrel{!}{=} 0$$

First Friedmann Equation

$$\frac{\frac{d}{dt}[\rho(a(t)l,t)]}{\rho(a(t)l,t)} = -3\frac{\dot{a}(t)}{a(t)}$$

Kinematic Condition

$$\frac{d}{dt}R(t) = u(R(t),t) \Rightarrow \frac{\frac{d}{dt}\left[t^{-\gamma}g(R(t),t)\right]}{g(R(t),t)} = -3\frac{t^{-\alpha}f(R(t),t)}{R(t)}$$

Power series in the similarity variable

$\rho(r,t) \sim t^{-\gamma} \sum_n^{\infty} \rho_n \zeta^n$ and $u(r,t) \sim t^{-\alpha} \sum_n^{\infty} u_n \zeta^n$

In the relevant space and time scale

- $\rho(r,t) \sim t^{-\gamma} A \zeta^{\kappa}$, where $\kappa \in \mathbb{R}^+$

We assume, that

$\rho(r,t) \sim t^{-\gamma} A \zeta^{\kappa}$, and $u(r,t) \sim t^{-\alpha} \sum_n^{8} u_n \zeta^n$

Non-rotating case: $\omega \to 0$ limit

Non-rotating:

$$u(r,t) \sim t^{-\alpha} \left(u_1 \zeta^1 + u_2 \zeta^2 \right)$$

Summarizing this,

Non-Rotating:	**Rotating:**
$\rho(r,t) \sim t^{-\gamma} A \zeta^{\kappa}$	$\rho(r,t) \sim t^{-\gamma} A \zeta^{\kappa}$
$u(r,t) \sim u_1 \zeta + u_2 \zeta^2$	$u(r,t) \sim t^{-\alpha} \sum_{k=0}^{8} \tilde{u}_k \zeta^k$

- Non-autonomous first-order non-linear differential equation

$$\kappa \dot{R}(t) + 3 u_2 t^{-(\alpha+2\beta)} [R(t)]^2 - \frac{1}{t}[\gamma + \kappa\beta] R(t) + 3 u_1 R(t) t^{-(\alpha+\beta)} = 0$$

General Solution for non-rotating case:

$$R(t) = \frac{u_1 t^{\beta+\gamma/\kappa} e^{-\frac{3u_1 t^{\mu}}{\mu\kappa}}}{3^{-\frac{\gamma}{\mu\kappa}} u_2 t^{\gamma/\kappa} \left(\frac{u_1 t^{\mu}}{\nu}\right)^{-\frac{\gamma}{\mu\kappa}} \Gamma\left(\frac{\gamma-\beta\kappa+\kappa}{\nu}, \frac{3u_1 t^{\mu}}{\nu}\right) - \mathcal{C}_1 u_1}$$

$$\mu := 1-(\alpha+\beta) \qquad \nu := \kappa - \beta\kappa$$

- The $\mathcal{C}_1$ is an integration constant
- Γ is the upper incomplete Gamma function.
- $(\alpha, \beta, \gamma, \delta)$ are known from the Sedov-Taylor Ansatz

$$R(t) = \frac{t}{\mathcal{C}_1 t^{\frac{3u_1-2}{\kappa}} + \frac{3u_2}{2-3u_1}}, \quad \text{where } \kappa = \frac{6}{7}$$

For the non-rotating case, the differential equation is

$$\kappa \dot{R}(t) - \frac{1}{t}[\gamma + \kappa\beta]R(t) + 3t^{-\alpha}\sum_{k=0}^{8} \tilde{u}_k \left(\frac{R(t)}{t^\beta}\right)^k = 0. \tag{1}$$

- It cannot be solved explicitly
- Hubble's law of expansion to determine the C_1 integration constant

$$\left.\frac{\dot{a}(t)}{a(t)}\right|_{t=t_0} = H_0, \text{ if } \quad a(t_0) = 1 \tag{2}$$

where $H_0 = 66.6^{+4.1}_{-3.3}$ km/s/Mpc[1] is the experimental value of the Hubble-constant.

[1]Kelly, P. L. et al. (2023) Science doi:10.1126/science.abh1322

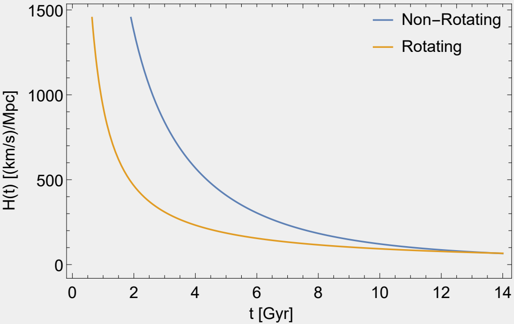


Figure: Analytical (Non-Rotating) and numerical (Rotating) solutions of the expansion rate of the Universe, the integration was started at $\zeta_0 = 0.001$, and the initial conditions of $f(\zeta_0) = 0.5$, $g(\zeta_0) = 0.008$, $h(\zeta_0) = 0$, and $h'(\zeta_0) = 1$ were used. The results match well with the data from literature[1].

[1] Xiaoyun Li, et al. J.HEP, Gravitation and Cosmology, Vol.8 No.1, 2022

Summary

- We used Sedov-Taylor-von Neumann ansatz to solve the Euler-Poisson equation
- We used polytropic EoS to describe the Dark Fluid
- Spherical symmetry and non-rotating/slow rotation
- Connection with the classical Newtonian Friedmann equation
- Expansion rate of the Universe

Thank you!

Questions?