The Tsallis-Thermometer as a QGP Indicator For Large And Small Collisional Systems

ELTE Particle Physics Seminar

Gábor Bíró

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Wigner Research Centre for **Physics**
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G. Bíró, **G.G. Barnaföldi**, **T.S. Biró**, **J. Phys. G**, 47.10 (2020), 105002.
G. Bíró, **G.G. Barnaföldi**, **K. Ürmössy**, **T.S. Biró**, **Á. Takács**, **Entropy**, 19(3), (2017), 88
G. Bíró, **G.G. Barnaföldi**, **T.S. Biró**, **K. Shen**, **EPJ Web Conf.**, 171, (2018), 14008
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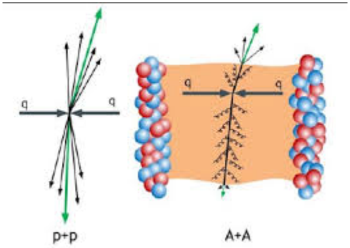
Motivation

Recap from **Róbert Vértesi: Scaling properties of jets in high-energy pp collisions:**

(s)QGP in A+A collisions:

- Hard probes: **jet quenching** (modification in the medium)
- Soft probes: **flow** (collective dynamics of the bulk)



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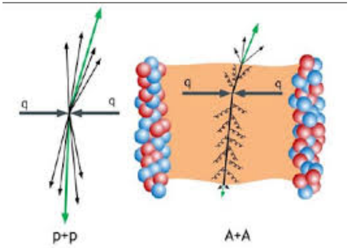
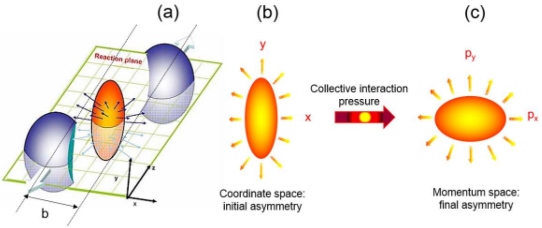
Collectivity in small systems:

- Observed at high-multiplicity events at LHC (**PRL 112, (2014), 082301**)
- QGP is **not necessary** for collectivity(?)

Jets in high-multiplicity events





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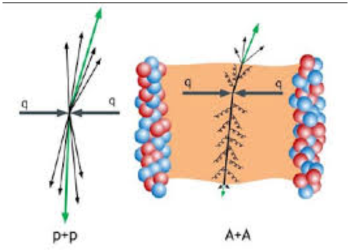
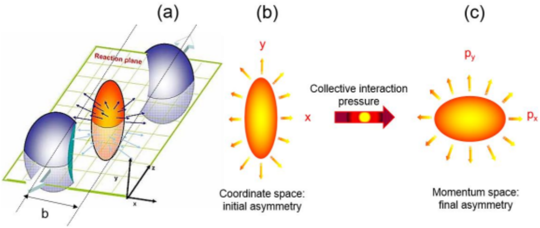
Jets in **high-multiplicity** events

"Basic question: Can we turn the QGP off?"

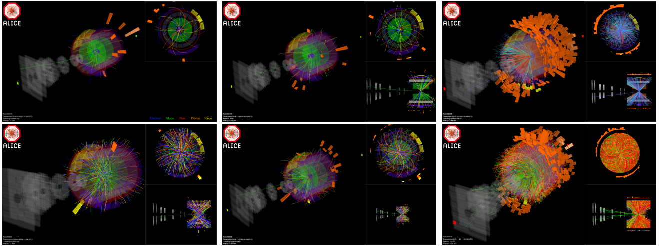


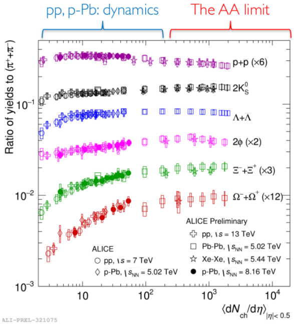


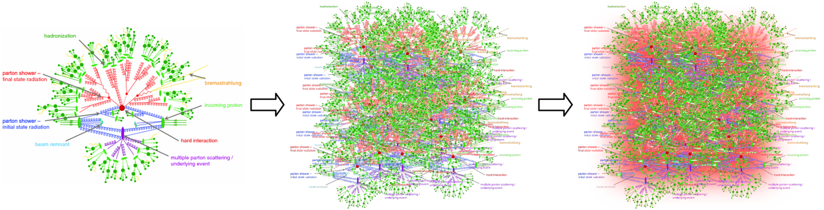
Experimental observable:

Ratio of identified hadrons in small to large systems…

…but what is small?







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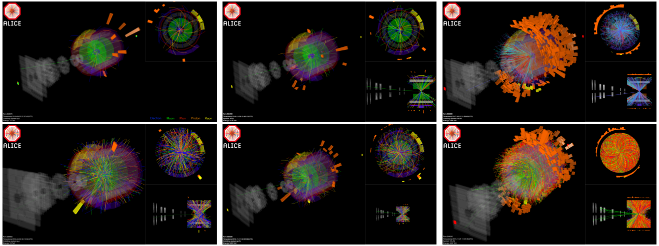

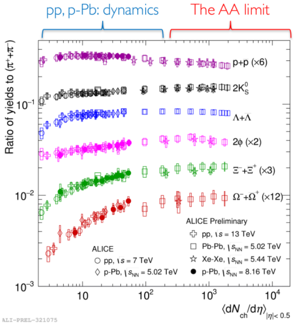

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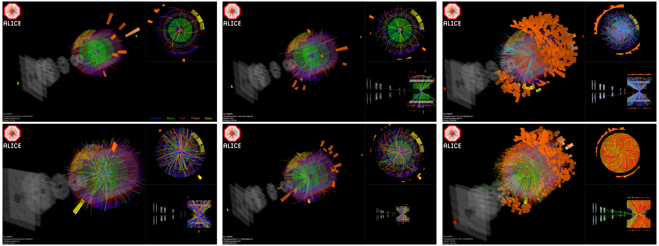
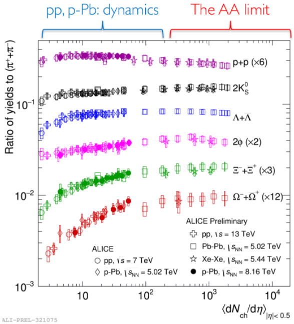
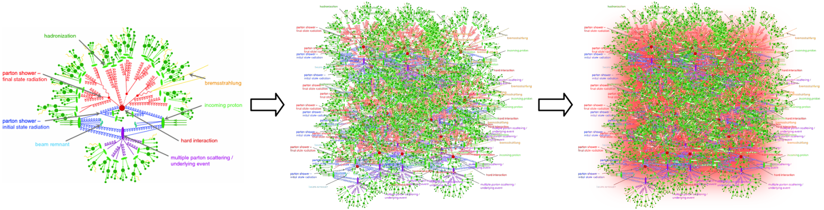
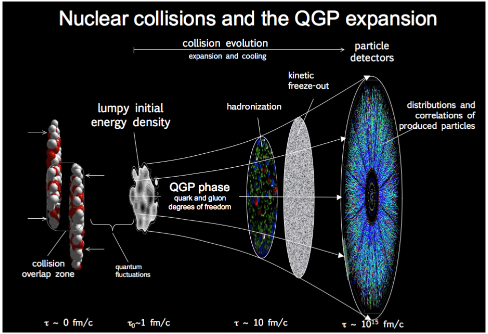
Where does the quark-gluon plasma start in **multiplicity**?

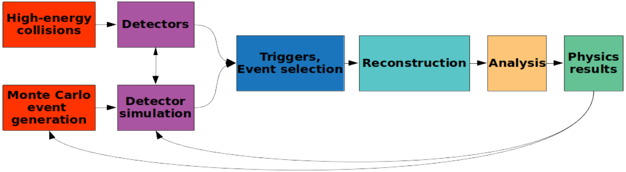






How to connect the theory (lattice QCD, ...) with the experiment (p_T spectra, multiplicity selection, ...)?





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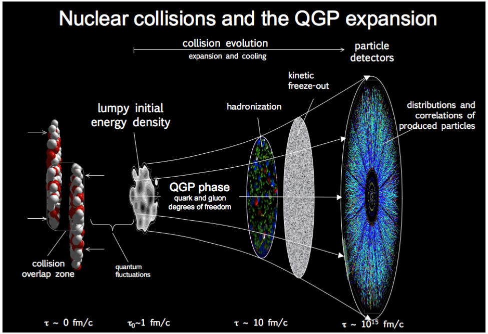
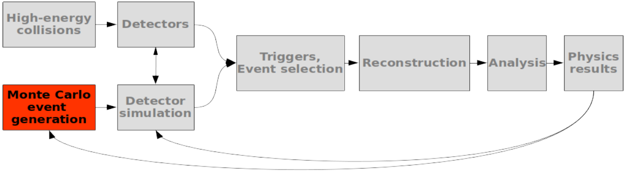


1. Develop theory to explain the observation (**HIJING++**)



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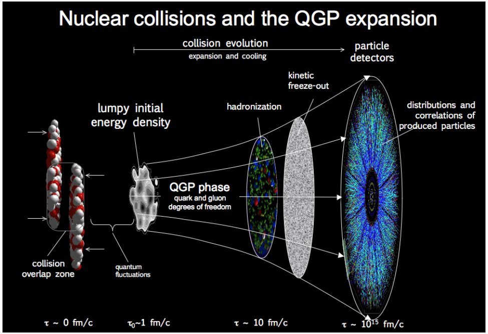
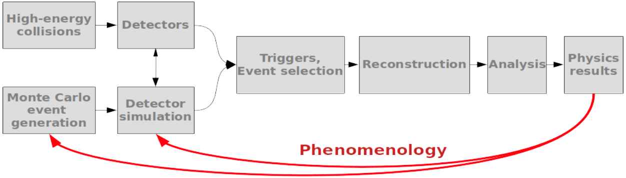


1. Develop theory to explain the observation (**HIJING++**)
2. Investigate the experimental results with **phenomenological** methods



Prelude

The Tsallis – Pareto-type fit functions describe the hadron spectra well –

any distribution form could work, but the physical considerations in the background may differ.

$$\frac{\mathrm{d}^2 N}{2\pi \mathrm{d}y p_T \mathrm{d}p_T} = \begin{cases} A\left(1+\frac{E}{T}(q-1)\right)^{-\frac{1}{q-1}}, \\ A m_T \left(1+\frac{E}{T}(q-1)\right)^{-\frac{1}{q-1}}, \\ A\frac{(n-1)(n-2)}{2\pi n T[nT+m(n-2)]}\left(1+\frac{E}{T}(q-1)\right)^{-\frac{1}{q-1}}, \\ \dots \end{cases}$$

where $n = 1/(q-1)$, and

$$E = \begin{cases} p_T, \\ m_T, \\ \gamma\left(m_T - v p_T\right), \\ \dots \end{cases}$$

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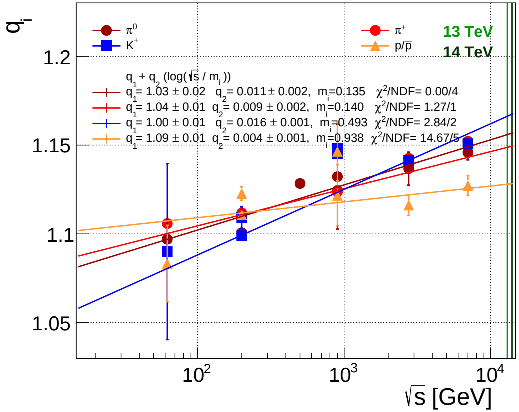
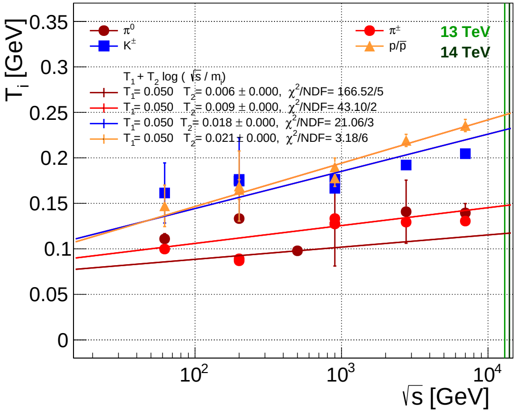
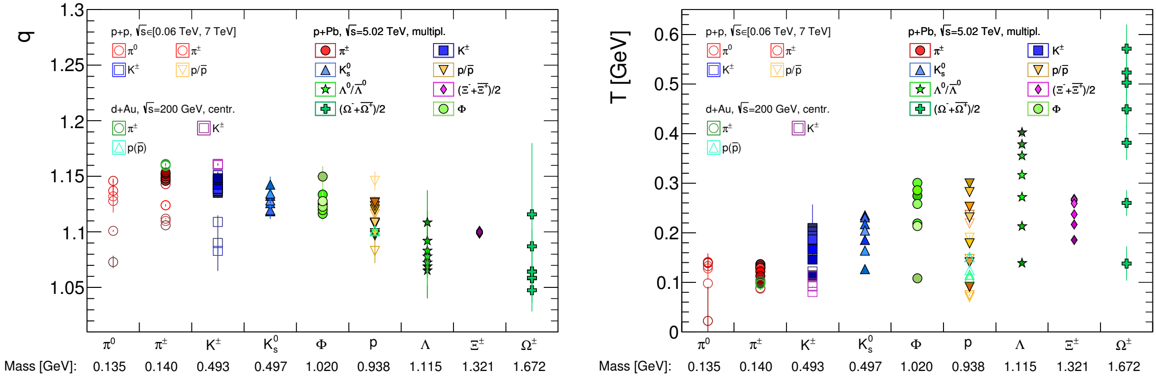
The fitted parameters depend on the center-of-mass energy:

$$q(\sqrt{s}) = q_1 + q_2 \log\left(\sqrt{s}/m\right) \qquad T(\sqrt{s}) = T_1 + T_2 \log\left(\sqrt{s}/m\right)$$





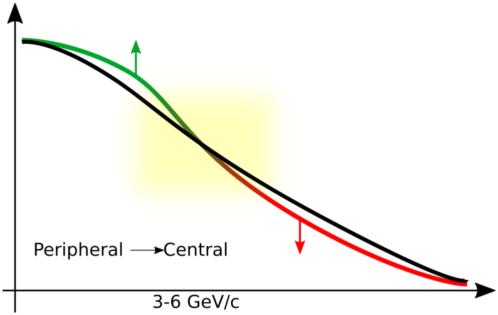
The fitted parameters present a strong mass hierarchy:



EPJ Web Conf. **171 (2018) 14008**

Goals

Non-extensive statistics – summary:



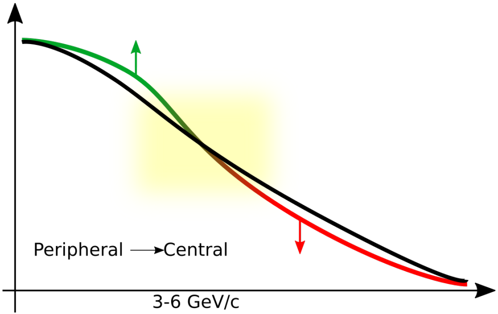
$$\frac{d^2N}{2\pi p_T dp_T dy} = A m_T \left[1 + \frac{q-1}{T}(m_T - m)\right]^{-\frac{q}{q-1}}$$

Non-extensive statistics – summary:

q-entropy:

$$S_q = \frac{1}{q-1}\left(1 - \sum_{i=1}^{W} p_i^q\right)$$

$$\lim_{q\to 1} S_q = S_{BG}$$



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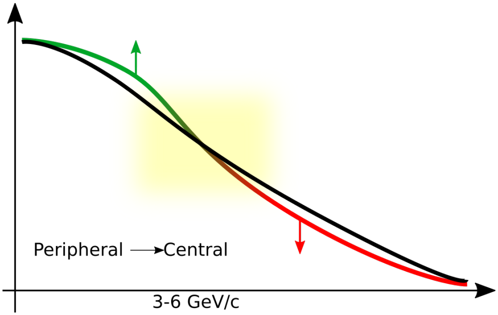
Thermodynamical consistency:

$$P = Ts + \mu n - \varepsilon$$

$$P = g\int \frac{\mathrm{d}^3 p}{(2\pi)^3}\, Tf \qquad s = g\int \frac{\mathrm{d}^3 p}{(2\pi)^3}\left[\frac{E-\mu}{T} f^q + f\right]$$

$$N = nV = gV\int \frac{\mathrm{d}^3 p}{(2\pi)^3} f^q \qquad \varepsilon = g\int \frac{\mathrm{d}^3 p}{(2\pi)^3}\, E f^q$$



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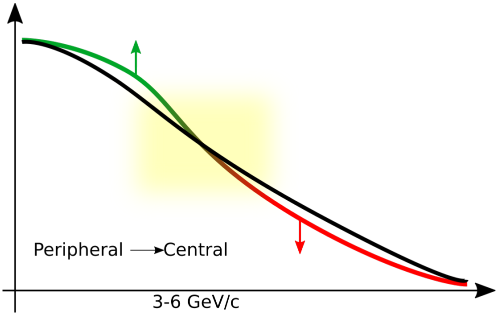
Final size effects:

$$T = \frac{E}{\langle n\rangle} \qquad q = 1 - \frac{1}{\langle n\rangle} + \frac{\Delta n^2}{\langle n\rangle^2}$$

$$T = E\left[\delta^2 - (q-1)\right] \qquad \frac{\Delta n^2}{\langle n\rangle^2} := \delta^2$$



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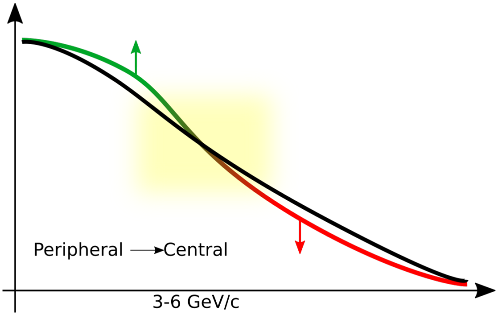

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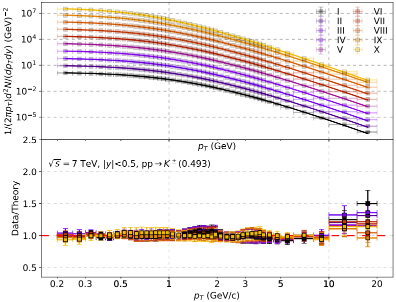

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Strong indication for multiplicity (system size) dependency:

$$\frac{\langle \mathrm{d}N_{ch}/\mathrm{d}\eta \rangle}{\langle N_{part} \rangle /2} \propto \begin{cases} s_{NN}^{0.15} & \text{for AA,} \\ s^{0.11} & \text{for pp.} \end{cases}$$

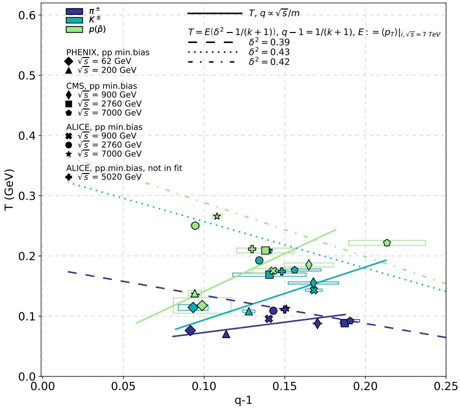
What is the relation with the earlier observations?

"Min. bias" pp: $\sim$ low multiplicity

1. solid lines: $\sqrt{s}$ dependency from earlier
2. $\sqrt{s}$ ~multiplicity~NBD

$$q \sim NBD \qquad (q-1 = 1/(k+1))$$

3. $T = E\left(\delta^2 - (q-1)\right)$, $E := \langle p_T \rangle|_{\sqrt{s}=7\,TeV}$



Entropy 16(12), (2014), 6497-6514

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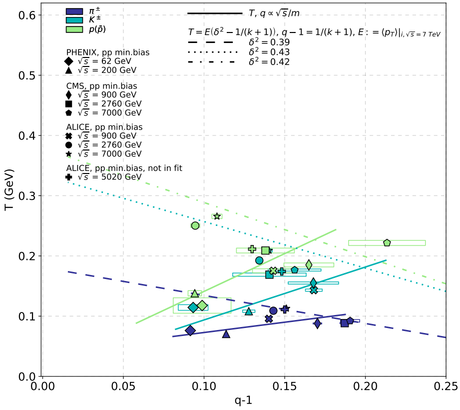


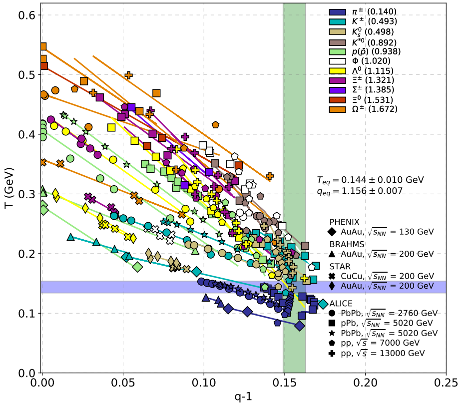
System size suggests an opposite trend!

Entropy 16(12), (2014), 6497-6514

The approach:

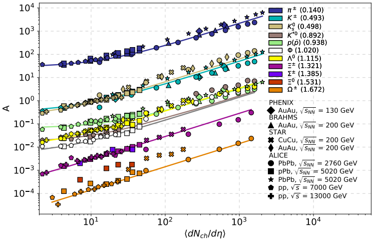
Map the thermodynamically consistent non-extensive parameter space of the available experimental data and compare it with theoretical QCD calculations

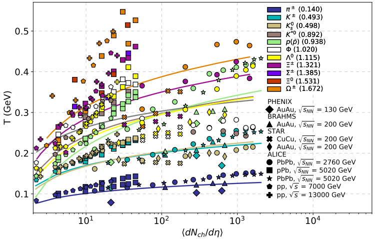
- 11 identified hadron species: from $\pi^{\pm}$ to Ω
- Various collision systems: proton-proton, proton-nucleus, nucleus-nucleus
- Wide range of multiplicities: $2.2 \leq \langle \mathrm{d}N_{ch}/\mathrm{d}\eta \rangle \leq 2047$
- Wide range of CM energies: $130 \leq \sqrt{s_{NN}} \leq 13000$ GeV
- **More than 30** published experimental datasets

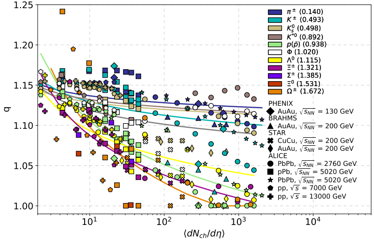


Goal: calibrate the Tsallis-thermometer

Results







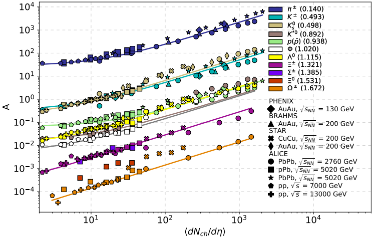
Parametrizations:

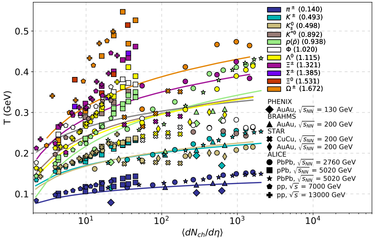
$$A = A_0 + A_1 \ln \frac{\sqrt{s_{NN}}}{m} + A_2 \langle \mathrm{d}N_{ch}/\mathrm{d}\eta \rangle$$

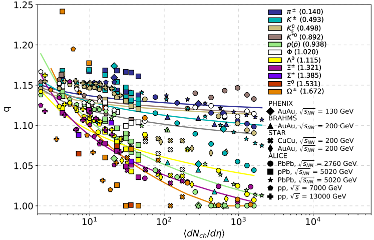
$$T = T_0 + T_1 \ln \frac{\sqrt{s_{NN}}}{m} + T_2 \ln \ln \langle \mathrm{d}N_{ch}/\mathrm{d}\eta \rangle$$

$$q = q_0 + q_1 \ln \frac{\sqrt{s_{NN}}}{m} + q_2 \ln \ln \langle \mathrm{d}N_{ch}/\mathrm{d}\eta \rangle$$

1. The **A**, **q** and **T** parameters characterize the collision
2. Strong **grouping:** $T_{eq} \approx 0.144$ GeV, $q_{eq} \approx 1.156$







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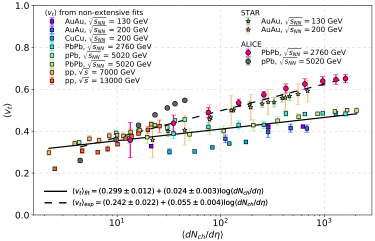
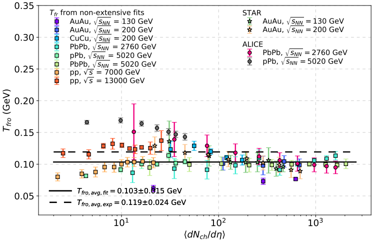




Radial flow:

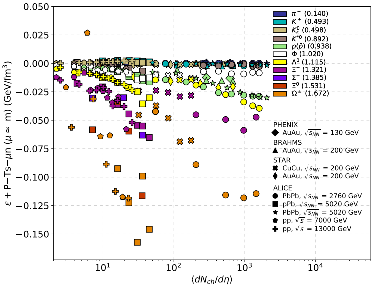
$$T = T_{fro} + m \left\langle u_t \right\rangle^2$$

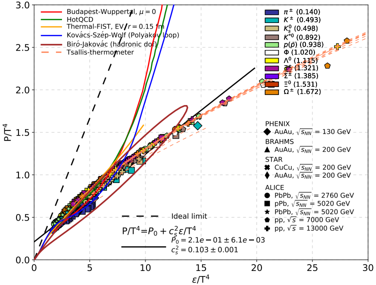
$$\left\langle v_t \right\rangle = \frac{\left\langle u_t \right\rangle}{\sqrt{1 + \left\langle u_t \right\rangle^2}}$$

1. The **A**, **q** and **T** parameters characterize the collision
2. Strong **grouping:** $T_{eq} \approx 0.144$ GeV, $q_{eq} \approx 1.156$
3. **Test:** results are comparable with experiments (**Phys. Rev. C 83 (2011), 064903**)

Thermodynamical consistency: ✓

$$P = Ts + \mu n - \varepsilon$$





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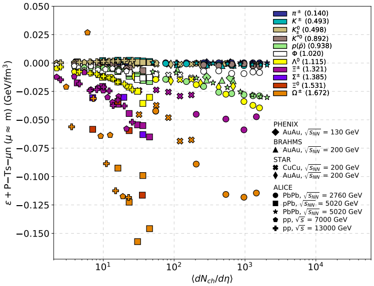
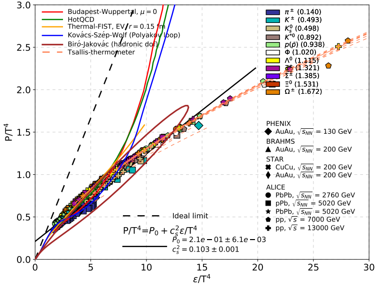
Comparison of the thermodynamical variables with theoretical calculations





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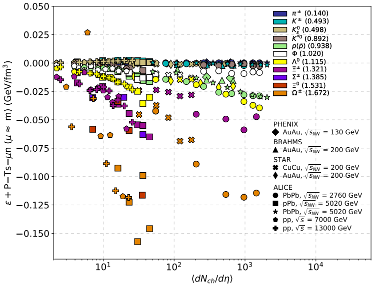

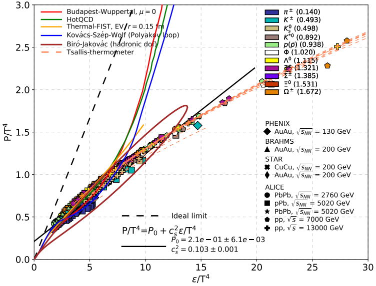

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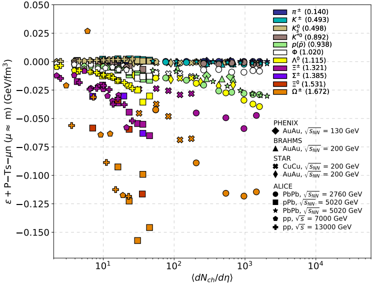

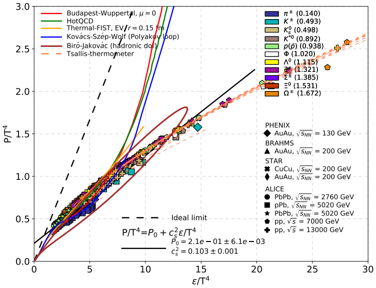

Hadronic dof ⇔ partonic dof

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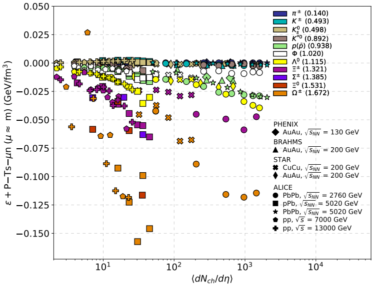

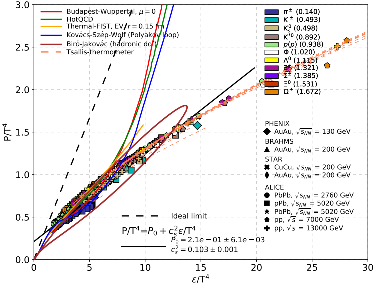

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Interpretation of the grouping phenomenon in the T - $(q-1)$ parameter space:

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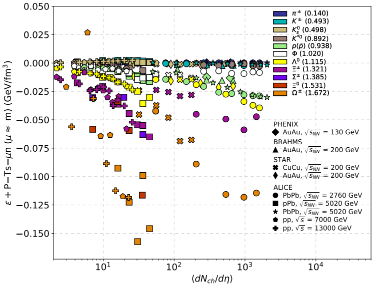

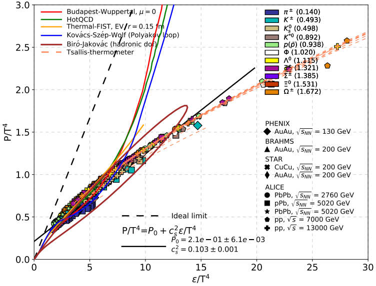

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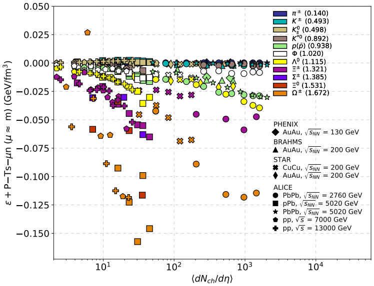

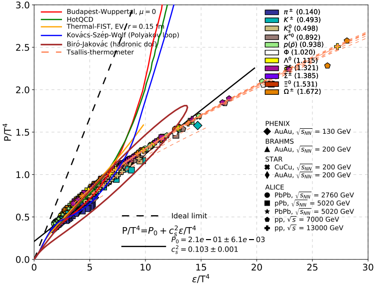

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1. Overlapping region with theoretical QCD calculations → **presence of hot QCD matter** just before the hadronization?
2. Hadron spectra of colliding systems with $T \approx 0.144$ GeV and $q \approx 1.156$: originates from a previous **quark-gluon plasma** state

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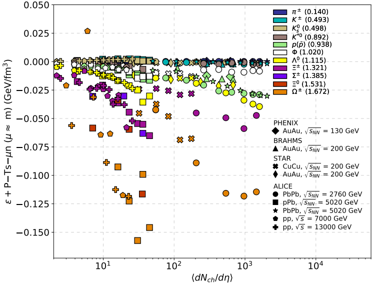

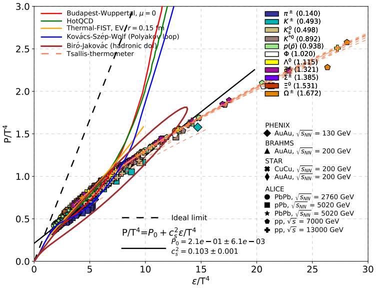

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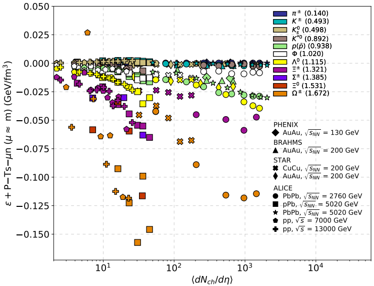

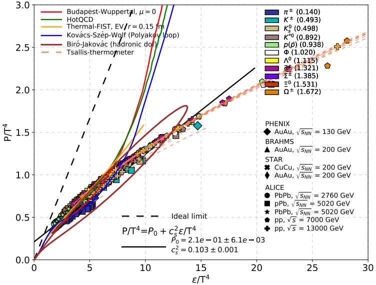

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With the **parametrizations:** $\sqrt{s}$ and $\langle \mathrm{d}N_{ch}/\mathrm{d}\eta \rangle$ regions:

- $\sqrt{s} \gtrsim 7000$ GeV: $\langle \mathrm{d}N_{ch}/\mathrm{d}\eta \rangle \gtrsim 130$
- $\sqrt{s} \gtrsim 13000$ GeV: $\langle \mathrm{d}N_{ch}/\mathrm{d}\eta \rangle \gtrsim 90$

Summary

- Consistent non-extensive analysis of a **very large set** of experimental data
- New results are in agreement with earlier studies
- $q \neq 1$ for all hadron spectra: dependency on the size of the collisional system through **multiplicity** fluctuations
- **Various checks** of the non-extensive framework
- Grouping of the **T** and **q** parameters
- **Comparison** with theoretical QCD calculations
- **Tsallis-thermometer:** final state hadrons may originate from a previously present strongly interacting QCD matter at event multiplicities as low as $\langle \mathrm{d}N_{ch}/\mathrm{d}\eta \rangle \sim 100$

Support

The research is supported by: OTKA K120660, K123815, K135515, THOR COST CA15213, Hungarian-Chinese 12 CN-1-2012-0016, MOST 2014DFG02050, 2019-2.1.11-TÉT-2019-00050 TéT, Wigner HAS-OBOR-CCNU, ÚNKP-17-3.

Thank you for your attention!

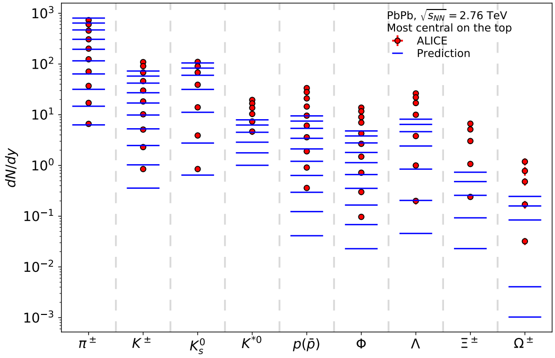
Backup

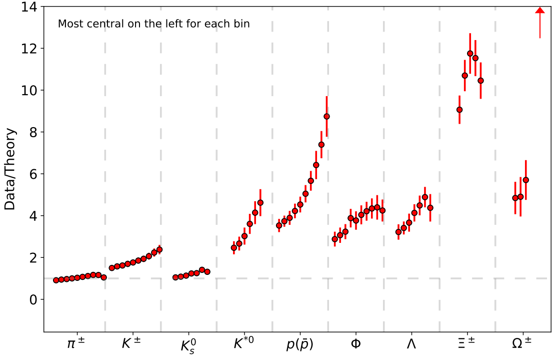
System, $\sqrt{s_{NN}}$ (GeV)	η or y	Hadron	Mult. classes	p_T range (GeV/c)
AuAu, 130	$\lvert\eta\rvert < 0{,}35$	$\pi^\pm$	3, [21,3; 622]	[0,25; 2,2]
		$K^\pm$		[0,45; 1,65]
		$p(\bar{p})$		[0,55; 3,42]
CuCu, 200	$\lvert y\rvert < 0{,}5$	K^0_s	5, [32; 175]	[0,5; 9,0]
		Λ^0		[0,5; 7,0]
		$\Xi^\pm$		[0,7; 6,0]
		$\Omega^\pm$		[1,0; 4,5]
		Φ	6, [24; 175]	[0,45; 4,5]
AuAu, 200	$\lvert y\rvert < 0{,}2$	$\pi^\pm$	3, [111; 680]	[0,2; 2,0]
		$K^\pm$		[0,4; 2,0]
		$p(\bar{p})$		[0,3; 3,0]
	$\lvert y\rvert < 0{,}5$	K^0_s	5, [27; 680]	[0,5; 9,0]
		Λ^0		[0,5; 8,0]
PbPb, 2760	$\lvert y\rvert < 0{,}5$	$\pi^\pm$	10, [13,4; 1601]	[0,1; 3,0]
		$K^\pm$		[0,2; 3,0]
		K^0_s	7, [55; 1601]	[0,4; 12,0]
		K^{*0}	6, [261; 1601]	[0,3; 20,0]
		$p(\bar{p})$		[0,3; 4,6]
		Λ^0		[0,6; 12,0]
		Φ		[0,5; 21,0]
		$\Xi^\pm$	5, [55; 1601]	[0,6; 8,0]
		$\Omega^\pm$		[1,2; 7,0]
pPb, 5020	$-0{,}5 < \lvert y\rvert < 0{,}0$	$\pi^\pm$	7, [4,3; 45]	[0,1; 20,0]
		$K^\pm$		[0,2; 20,0]
		K^{*0}	5, [4,3; 45]	[0,0; 16,0]
		$p(\bar{p})$		[0,35; 20,0]
		Φ		[0,4; 20,0]
		Ξ^0	4, [7,1; 35,6]	[0,8; 8,0]

System, $\sqrt{s_{NN}}$ (GeV)	η or y	Hadron	Mult. classes	p_T range (GeV/c)
pPb, 5020	$-0{,}5 < \lvert y\rvert < 0{,}0$	$\pi^\pm$	7, [4,3; 45]	[0,1; 20,0]
		$\Sigma^\pm$	3, [7,1; 35,6]	[1,0; 6,0]
		$\Xi^\pm$	7, [4,3; 45]	[0,6; 7,2]
		$\Omega^\pm$		[0,8; 5,0]
	$0{,}0 < \lvert y\rvert < 0{,}5$	$\pi^\pm$	7, [4,3; 45]	[0,1; 3,0]
		$K^\pm$		[0,2; 2,4]
		K^0_s		[0,0; 8,0]
		$p(\bar{p})$		[0,3; 4,0]
		Λ^0		[0,6; 8,0]
PbPb, 5020	$\lvert y\rvert < 0{,}5$	$\pi^\pm$	10, [19,5; 2047]	[0,1; 10,0]
		$K^\pm$		[0,1; 10,0]
		$p(\bar{p})$		[0,1; 10,0]
pp, 7000	$\lvert y\rvert < 0{,}5$	$\pi^\pm$	10, [2,2; 21,3]	[0,1; 20,0]
		$K^\pm$		[0,2; 20,0]
		K^0_s	10, [2,2; 21,3]	[0,0; 12,0]
		K^{*0}	9, [2,2; 21,3]	[0,0; 10,0]
		$p(\bar{p})$	10, [2,2; 21,3]	[0,3; 20,0]
		Φ	9, [2,2; 21,3]	[0,4; 10,0]
		Λ^0	10, [2,2; 21,3]	[0,4; 8,0]
		$\Xi^\pm$		[0,6; 6,5]
		$\Omega^\pm$	5, [2,2; 21,3]	[0,9; 5,5]
pp, 13000	$\lvert y\rvert < 0{,}5$	K^0_s	10, [2,52; 25,72]	[0,0; 12,0]
		Λ^0		[0,4; 8,0]
		$\Xi^\pm$		[0,6; 6,5]
		$\Omega^\pm$	5, [3,58; 22,8]	[0,9; 5,5]

Hadron yields from the parametrizations

$$\left.\frac{\mathrm{d}N}{dy}\right|_{y=0} = 2\pi A\,T\left[\frac{(2-q)m^2+2mT+2T^2}{(2-q)(3-2q)}\right]\left[1+\frac{q-1}{T}m\right]^{-\frac{1}{q-1}}$$





Thermodynamical quantities